SCHOOL OF EDUCATION
UNIVERSITY OF MICHIGAN

# GeT: The News!

Volume 1, Issue 1
Fall 2019

## Did You GeT: The News?

by Pat Herbst

Welcome to the first issue of *GeT: The News*—the newsletter for GeT: A Pencil! The idea with this newsletter is to deepen our community bonds and share ideas. We plan to publish it three times a year and to include not only news from the GRIP Lab about our GeT Support project, but also essays by and notices from members of the GeT: A Pencil community. In this issue, we discuss the advantages and disadvantages of the Euclidean Archetype (pg. 1), examine tasks related to angle bisectors (pg. 2), and investigate a potential source of errors in triangle construction (pg. 4). I also encourage you to look at the list of events on page 2—there are some upcoming conferences that we're very excited about and we'd love to see you there.

Hopefully, you have been involved this fall in some activities within the GeT: A Pencil community. We thought we'd give you an overview of the big picture. We have been running a cycle of Working Seminars since early October. Since then, we have held two seminars each month. Each seminar starts with a presentation by one of us (so far presenters included Amanda Milewski, Pat Herbst, Sharon Vestal, Shawnda Smith, and Nat Miller) and ends with questions for asynchronous discussion. If you have to miss the seminars, it's worth noting that we record and post them in GeT: A Pencil, so you can catch up and still join in on the asynchronous conversation (use the Canvas Modules feature to find them). The working discussions have featured themes including teaching geometry with technology, teaching proof, instructional situations, and mathematical practice. During the first working discussion, Amanda Milewski spoke about key stakeholders we might need to hear from in order to learn about the needs and impact of Geometry courses for teachers. She shared that we have been interviewing administrators and department chairs at the K-12 level to get their input. In another seminar in January, you'll have a chance to hear more about what we are learning from those interviews.

In addition to the working seminars, two working groups have been assembling within GeT: A Pencil: **Transformations** and **Teaching GeT**. The Transformations Working Group meets every other week. In their meetings during the Fall, they have been discussing goals in teaching transformation geometry. They are collecting possible starting axioms for a transformations-based class, and are sharing classroom activities and course notes. Julia St. Goar from Merrimack College manages the group. They have a Google Doc to which everyone contributes during the meeting time. This

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*Contributed Essay*

## A GeT Course "Classic": The Euclidean Archetype

by Steve Cohen, Nat Miller, and Steve Szydlik

We are all members of the Euclidean Archetype workgroup. As we summarized in our report, a GeT course organized around the Euclidean archetype will focus on the axiomatic development of fundamental principles of geometry. Informed by the spirit and organization of Euclid's Elements, this course emphasizes mathematical precision, rigorous proof, and clear communication. We have all taught geometry with varying amounts of experience. We agree on many goals that a course should have but each of us prefers a different balancing of the ingredients. What follows is our discussion of the essential elements and the plusses and minuses of teaching a class using the Euclidean Archetype.

**THE ESSENTIAL COMPONENTS**

**SS:** The Euclidean archetype centers on axiom systems, and any GeT course following this framework should emphasize that structure: precise language, identification of agreed-upon undefined terms and axioms, and the development of the theorems of geometry from those foundations. A worthy highlight of this course is the

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*Did You Get: The News? - continued from pg. 1*
document also has a list of textbooks an instructor can use.

The Teaching GeT Working Group's aim is to document the essential student learning objectives of a GeT course: are there common goals for the course, perhaps independent from the choices instructors make when they decide what type of geometry course to teach? This group meets once a month and collaborate to produce documents and resources for people who are teaching or will be teaching a GeT course. Nat Miller from the University of Northern Colorado leads this group. Each month, members of the group independently produce an assigned task, which Nat then synthesizes into a single document to determine common interests. Their most recent assignment was to come up with essential understandings for a GeT course (i.e., what are student learning objectives that instructors believe should be covered in any Geometry for Teachers class?). His group is building on the archetype work from from last year's working group on the knowledge of geometry needed for teaching, which documents the various kinds of GeT classes in practice.

As all of this is happening in GeT: A Pencil, several of you have been teaching the GeT course. Your students have completed the MKT-G pre-test and are getting ready to complete the post-test. Several of you have also completed instructional logs, helping the community to document teaching practices used in the course. We are eager to share some gleanings of the aggregate once we have a critical mass from which to report. Like in previous semesters, we will soon be sending along the end-of-course questionnaire. And we look forward to working with those of you who will be teaching in the Winter and Spring. One change we will be making this year in regard to our aggregate reports is that we'll hold off until May to give you an aggregate report of MKT-G growth. We'll be presenting at AERA this year the results of the analysis of last year's growth. Thanks to your collaboration, we have administered the MKT-G test to 222 GeT students, about half of whom were intending to be teachers. We think the results of our analysis will be interesting to you as a GeT instructor: we found that while those GeT students intending to be teachers had lower pretest scores than the other GeT students, their scores grew significantly more during the time of the GeT course. And while we can't attribute causality to the course or to their career orientation, it is nice to know that some improvement of capacity to teach high school geometry is observable! We are hopeful we'll be able to add to this information in our May MKT-G report.

I hope you enjoy this newsletter. Have a great holiday season—filled with well-deserved rest and relaxation with your loved ones.

*Matt Park and Chandler Brown contributed to this note.*

*GeT Task*

## Angle Bisectors by Matt Park

> *There are good reasons why the theorems should all be easy and the definitions hard.*
> *-Michael Spivak*

In this article, we look at a selection of tasks related to the angle bisectors of a quadrilateral and discuss their potential function in GeT courses. An instructor may choose to begin an exploration of the properties of the angle bisectors by looking at the quadrilateral formed by the lines in question. But this already provides an opportunity for students to engage in thinking much like a math researcher, as well as to discover the properties of these lines. Consider the following task:

> Consider the four angle bisectors of a quadrilateral $Q_1$. How many times can at least two bisectors intersect? What rules can consistently choose four points to define a second quadrilateral $Q_2$? When $Q_2$ exists, what is its area and when is it 0? Deduce other properties of $Q_2$?

At the heart of this task is the request for the student to formulate a definition. The angle bisectors of a quadrilateral could form as many as 6 distinct intersections, and it is not trivial to determine which ones are "natural" choices to determine a second quadrilateral. While the "correct" definition is to think of angle bisectors as rays oriented inward instead of lines, the open-endedness of the question nevertheless allows students to perhaps build their own justification (sound or not). An instructor could continue the class by asking students to compare the definitions they've created, so they can see the creative and interpretive nature of mathematics.

The next task is designed to give students a greater sense that definitions are not chosen arbitrarily, but because of their power in proving theorems. Once the class agrees to define the "induced quadrilateral by the angle bisectors" as those with vertices of the intersection of the rays, consider the following task:

> Group task: Write definitions of a square, rectangle, rhombus, and kite to make the following theorems true when the area of the induced quadrilateral is 0.
> - For a quadrilateral Q, Q is a parallelogram if and only if the angle bisectors of Q form a rectangle.
> - For a quadrilateral R, R is a rectangle if and only if the angle bisectors of R form a square.

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## GeT to know the community

Did you win a grant? Get promoted? Have a baby? Buy a house? We would love to feature your news, whether professional or personal! Email us at **GRIP@umich.edu**.

*Euclidean Archetype - continued from pg. 1*
independence of the parallel postulate. This requires some work, including a careful development of the concepts of models and independence and an exploration of alternative axiom systems for Euclidean geometry (including Euclid's axioms and some other modern system).

**SC:** I agree. I would add that the structure naturally leads to an emphasis on proof writing. I find it useful to spend some time in a simpler axiom system such as an incidence geometry to enable students to practice writing proofs with fewer subtleties and issues.

**NM:** I think there are two key components here, that don't necessarily have to be combined, but often are. This is sometimes referred to as the Euclidean Axiomatic archetype, and the two components are Euclidean Geometry and an axiomatic approach. You could have a course focused purely on Euclidean Geometry; you could have a purely axiomatic geometry course; and putting them together, you could have an axiomatic geometry trying to get at the main ideas of Euclidean geometry. There are certainly courses that mostly do one of these without the other. For example, some books focus on explorations of Euclidean geometry using dynamic geometry software without an axiomatic approach. On the other hand, some completely axiomatic courses don't get very far into Euclidean geometry because it takes so long to prove elementary facts about incidence geometry and betweenness proceeding carefully from elementary axioms. Probably to be considered part of the Euclidean Axiomatic archetype, you need to explore some of both. Getting to both probably requires that we broaden both pieces, though. As SS notes, we will want to talk about models and independence, which will require us to work, at least a bit, with some non-Euclidean geometries; and to get to the interesting parts of Euclidean geometry, we will probably have to move away from the idea of proving absolutely everything from a purely axiomatic standpoint.

**OTHER TOPICS TO INCLUDE**

**SS:** Proving the independence of the parallel postulate opens up the world of non-Euclidean geometry, and exploring the seemingly strange world of hyperbolic geometry is a natural branching off point for this archetype. It provides students with an alternative axiom system to consider and by developing its major theorems students gain a stronger understanding of the more familiar Euclidean world. The archetype also provides an opportunity to study Euclidean straightedge and compass constructions. Careful development of these tools provides significant payoff if the instructor chooses to investigate models of hyperbolic geometry in some detail. Dynamic geometry software can be a powerful tool in this investigation.

**SC:** Compass and straightedge constructions are foundational in Euclidean geometry. Students can use

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*Angle Bisectors - continued from pg. 2*
Traditional treatment of these theorems require that Q is not a rhombus, nor R a square. So this is an especially natural task to pose after an instructor has presented the theorems in the traditional manner. Not only does it allow the students to make the statements more elegant, but it also shows them that a theorem is occasionally improved not through an improved proof, but through a generalization of the definitions. So with these tasks, a GeT instructor can guide their students not only through Euclidean geometry but also through research-level mathematical thinking. A nontrivial amount of time spent in research is useful not necessarily in obtaining results about a mathematical object, but in pinpointing the abstract properties of the object that necessitate the results in question. Through these tasks, a GeT instructor can create this environment of creating definitions: an undertaking that, as Spivak indicates, can be quite difficult.

*Matt Park is a research assistant in the GRIP Lab.*

## What's happening?

**December**

3 (Tuesday): Working Seminar - Matt Windsor
12-1pm EST, online

13 (Friday): Working Seminar - Steve Boyce
2-3pm EST, online

**January**

17 (Friday): Joint Math Meeting, Denver, CO
*We will be presenting a progress report of the GeT Support project in the afternoon (details TBA). Community members welcome.*

**February**

27 (Thursday): GeT: A Pencil Working Group at the Research in Undergraduate Mathematics Education (RUME) 2020 Conference, Boston, MA. 8am-12pm, location TBA within the Revere Hotel, Boston Common
*We encourage all GeT: A Pencil members to join us! Please note that because this is the morning of the first day of the conference, attending the working group will likely require you to travel to Boston on Wednesday the 26th.*

**Save the Date**

April: we will be presenting results from our analysis of MKT-G growth at the American Educational Research Association (AERA) conference in San Francisco, CA

To list an event in an upcoming newsletter, email us at **GRIP@umich.edu**.

*Euclidean Archetype - continued from pg. 3*
these to make conjectures, prove theorems, and develop geometric intuition. Students can also consider models where various axioms fail to hold, such as geometry on the sphere, or on the Cartesian plane using the taxicab metric to measure distance.

**NM:** I agree with all of these ideas. Spherical geometry is also natural to look at in the context of parallel lines– with spherical, Euclidean, and hyperbolic geometry, we have cases with no, one, and more than one line(s) through a point parallel to a given point. I think spherical geometry is more accessible to students since they already know what a sphere is. There is also a sense in which spherical, Euclidean, and hyperbolic geometry are the building blocks for all 2 dimensional geometries. There is dynamic geometry software for each of these, and I also like to have students work with physical models.

**ADVANTAGES**

**SS:** I love the structure of this archetype. Building geometry from a set of axioms and undefined terms allows students to see a logical development of the subject. Even a short exploration of the Elements gives students an appreciation for the monumental achievement of Euclid while helping them recognize the need for precise language and rigorous proof. In addition to focusing on strengthening students' logical reasoning abilities, the archetype also offers natural opportunities to build in a historical examination of geometers, from Thales to Saccheri to Bolyai to Riemann to Hilbert, as well as many others in between. I believe that a strong foundation in the axiomatic structure of geometry is an essential component of the preparation of future teachers of the subject.

**SC:** Euclidean Geometry has long been a model of deductive reasoning and teaching students to write proofs. Teaching it also presents a great opportunity to incorporate the humanities (art, history, western civ.) into the math curriculum. Most exciting part of teaching it for me was following the long and technical journey through Neutral geometry not allowing students to assume familiar results such as 180 degrees in a triangle. When, finally we bring in the Euclidean Parallel postulate, the parallel projection theorem, similar triangles, the Pythagorean Theorem, and trigonometry immediately enrich the study. Finally, it is natural to discuss practical applications.

**NM:** Geometry has long been a place in the mathematics curriculum where logic is discussed in a mathematical setting. I don't think there is a better setting than a geometry class to get students thinking about the roles of axioms, definitions, and theorems, and to start thinking about metamathematical ideas about when statements are unprovable in a given system.

**DRAWBACKS**

**SS:** With its emphasis on an axiomatic development of geometry, this archetype does not as naturally lend itself to applications or pedagogical conversations as some other archetypes might. Moreover, Euclid's axioms have little to say about geometric transformations, an important component of the Common Core State Standards for Mathematics. However, these topics could be included with careful planning by the instructor.

**SC:** Preservice teachers need additional perspectives, extended time with transformational geometry, and opportunities to do the kind of exploration emphasized in the common core. It is possible but much more challenging to include these features in a Euclidean course.

**NM:** One big drawback of a purely axiomatic approach is that there isn't an axiomatization of Euclidean geometry that is fully complete and rigorous that is at the appropriate level for most undergraduates. If we use something like Hilbert's axiomatization, we end up spending a lot of time giving fairly technical proofs of trivial results. Actually, Euclid's treatment is still one that is at about the right level for most students, but it does make some unstated assumptions. The other piece that this approach usually leaves out is the opportunity for students to explore and make conjectures before trying to prove them, which is another giant piece of doing mathematics that geometry courses are especially well suited for. That's why I tend to structure my courses around the experiencing geometry archetype, but for all the reasons we have discussed, I almost always include a section of the course structured around the axiomatic Euclidean archetype. One way to do this is to spend several weeks in the middle of the course having students prove basic theorems of neutral geometry from a simple four axiom system.

*Steve Cohen is Associate Professor of Mathematics at Roosevelt University.*

*Nat Miller is Professor of Mathematical Sciences at the University of Northern Colorado.*

*Steve Szydlik is Professor of Mathematics at the University of Wisconsin Oshkosh.*

## Contribute an essay

We welcome contributions from members of the GeT: A Pencil community! Activities you tried in class, things you observed your students do, reflections on your experience teaching, thoughts on what the GeT course should include. . . any of these and others would be fair game to write about. Consider the length of the articles in this issue as examples of how long your piece could be. To pitch your idea, email us at **GRIP@umich.edu**.

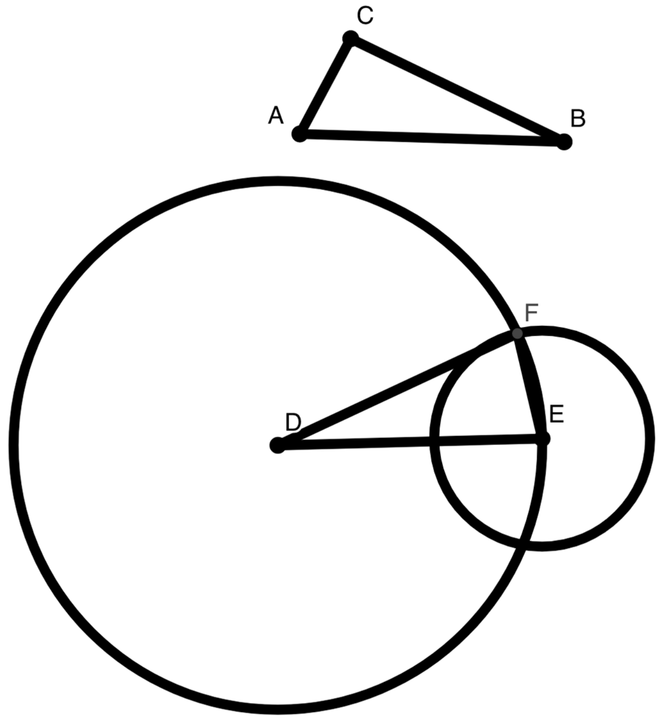
*GeT Activity*

# An Odd Copy of a Triangle: Where do student errors come from?

by Pat Herbst

The mathematics education literature on student errors has documented how sometimes what students learn can be overgeneralized as they solve other problems and can even occasion errors. I was reminded of this as I puzzled over something I observed some Geometry for Teachers' students doing as they worked on the problem of constructing a triangle congruent to a given triangle.

Students were asked to create a triangle DEF whose sides would be congruent with those of a triangle ABC which was given. One student, "Angie," produced a construction like the one below.




Angie's construction was incorrect. She constructed $\overline{DE}$ to be congruent to $\overline{AB}$ and $\overline{EF}$ to be congruent to $\overline{AC}$, which was promising. If by the notation (X, $\overline{YZ}$) we mean the circle of center X and radius $\overline{YZ}$, we can observe the following. Angie found Point F to be the intersection of the two circles (D, $\overline{AB}$) and (E, $\overline{AC}$), which meant that F belongs in circle (D, $\overline{AB}$) and hence $\overline{DF}$ would be congruent to $\overline{DE}$. Why Angie thought that $\overline{DF}$ would also be congruent to $\overline{CB}$ was not apparent to me.

Then I realized that Angie and her classmates had just learned how to construct an equilateral triangle with straightedge and compass. In this construction, students had learned what Euclid does at the very beginning of Book 1. Euclid creates two circles, using the extremes of a given segment as centers and using the segment as radius. The third vertex of the equilateral triangle is found at the intersection of these two circles. Other than the fact that (E, $\overline{AC}$) had a different radius, the procedure was very similar. There was also the alluring presence of a new point of intersection—two, in fact. If those points were not meant to be the points sought, what could one do with them?

Of course, the correct point F would need to be found by constructing circle (D, $\overline{BC}$) and intersecting this circle with circle (E, $\overline{AC}$). One could nudge students in that direction by, for example, asking them where to find all the points at a distance $\overline{BC}$ from D. Hopefully, that question would get them to think that there is a third circle that needs to be constructed.

## GeT Support

Sponsored by NSF DUE-1725837. All opinions are those of the authors and do not necessarily represent the views of the National Science Foundation or the University of Michigan.

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*GeT Support is housed in the GRIP Lab at the University of Michigan*

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