



GeT: The News!

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Reflecting on What Our Community Has Accomplished

by Pat Herbst

This new edition of GeT: The News! arrives on the brink of the second anniversary of the COVID 19 pandemic—not that anybody wants to celebrate that birthday. However, I do celebrate that, in the midst of this enduring disruption, our community has continued to be involved in several joint projects. As we took stock of our 2021 activities when we met earlier in January, I realized how much the work of this community means to me.

The completion of the writing of the chapter titled “A cross-institutional faculty online learning community: Community-guided faculty development in teaching college geometry for teachers” was a particular highlight for me. It brought attention to something I have always believed in—that professionals can learn from each other as they collaborate on tasks that pertain to their usual work. In one way or another, this idea has been behind most of the GRIP’s efforts to create opportunities for professional learning. Rather than assuming that we know what instruction should look like and that we should tell that to instructors, we prefer to create work environments in which instructors can collaborate solving problems of practice and envisioning improvements. The text of the chapter, as well as the work that went into getting the chapter written, suggest that, at least with the GeT: A Pencil community, our approach to professional learning can work. I think of it as *soft professional development*; we may not know ahead of time what we are going to learn, but the actual learning that eventually takes place feels more authentic than what one usually gets in professional development workshops that follow a set agenda. The group-generated projects, namely the identification and elaboration of student learning objectives and the development of activities to teach geometry via transformations, have that kind of authenticity to them; yet the most important learning that has been taking place is of a social nature. We have learned to work together and complement each other, accepting the talents and the quirks that everybody brings.

When we started the GeT: A Pencil community three years ago, we brought together a group of instructors of geometry courses for teachers under the premise that an inter-institutional community might fulfill a need; because within each mathematics

department there are usually few instructors who teach geometry, it is unlikely that learning communities specific to the subject matter can be organized within institutions. At the 2018 conference, we noticed how much of a divide there seemed to be between those trained in mathematics and those trained in mathematics education and suspected that collaboration might take effort. While our initiative to bring mathematicians and mathematics educators together is only one among many, it is good to see that after three years of working together, the community has not only endured but also grown in numbers, as well as in collective accomplishments.

At our meeting in early January one comment suggested that GeT: A Pencil has functioned as kind of a virtual mathematics department. I loved the metaphor. In fact, I wondered if it might suggest a path forward in the development of expertise for the mathematical preparation of teachers. Might similar communities be developed for instructors who teach mathematics classes for elementary teachers, for example?

The pandemic has touched our lives in many ways, but at least in regard to the work of GeT: A Pencil, it has not prevented us from continuing to learn from each other. Instead, as much of our regular work has also been happening through video conferencing, it seems that the difference between inter-institutional and intra-institutional collaboration has become less salient.

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Reflections on Teaching about Symmetry and Mutuality

by Steven Boyce

I remember well the day I first met Ms. Trump. I had been hired to teach mathematics at Pulaski County High School in Southwestern Virginia. My arrival in August 2003 coincided with the retirement of the mathematics department chair, and Diana Trump was his replacement. She explained that with their seniority system, I was assigned to teach three courses of geometry. To my surprise, she explained that she was the only teacher in the department who enjoyed teaching geometry; most everyone preferred teaching algebra. Although I had not completed student teaching in geometry, I felt prepared to teach the course. I had taken three college courses focused on Geometry for Teachers: an independent study of Euclid's Elements as a sophomore as well as senior-level College Geometry and Capstone courses.

Although I had earned my teaching certification at a university only 30 minutes away, teaching that first year was a culture shock. I endeavored to find ways to connect geometry to my students' lives, as many were not intending to attend college. I recall one activity from the Glencoe text we were using involved the identification of symmetries in automobile company logos. I was surprised that so many of the students in my classes knew the corresponding brands. It may have been due to the popularity of automobile racing in the community or that the largest employer in the county was a Volvo truck plant. I explained that some of the brands were actually owned by the same company (e.g., Honda and Acura). In addition to asking them to identify the symmetries in the decals, I tasked them to consider patterns in the symmetries of decals owned by the same company and why the company might or might not want a consumer to form associations between brands. My recollection is that no other lessons I taught that year had as engaged and active student participation.

Many things have changed in the past 19 years. I had introduced symmetries in my high school geometry class toward the end of the course, which was how it was organized in the Glencoe text. The role of transformations and the development of transformational reasoning has since become central to curricular standards, as evidenced by the Common Core State Standards (CCSS-M). The CCSS-M defines two figures to be congruent if there exists an isometry mapping one figure onto the other (rather than congruence of an image and its preimage being a property of an isometry). My high school students had categorized decals by the type of symmetry: point (half-turn or 180-degree rotation), line (reflectional), rotational (including point), or "none". Categorizing symmetries this way is mostly excluded from the geometry Standards. An exception is a third-grade standard, [CCSS.MATH.CONTENT.4.G.A.3](#): "Recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. Identify line-symmetric figures and draw lines of symmetry."

Another change is that I am now teaching College Geometry in Portland, Oregon. Entering my GeT course, most of my students (many of whom are older than traditional college age) have learned about symmetries as properties of figures. This may be because transformational reasoning has only recently become more central in geometry. A learning goal I have for my students is expanding their understanding of a plane symmetry as a type of isometry. Based on my previous experiences teaching College Geometry, I had thought of three stages in GeT students' understanding of a symmetry as a transformation, marked by changes in the domain of consideration. In the first stage, a symmetry is a transformation that acts on a proper subset of a figure. A student with this conceptualization might explain that a reflection that maps the left-half of a figure onto the right-half of the figure is a symmetry. With the second conceptualization, the student considers an entire figure as the domain. The student would consider that the same single reflection also maps the right half of the figure onto the left half. For the third conceptualization, the student considers that as an isometry, a symmetry is a bijective mapping of the entire plane (and thus they would not just consider the figure of interest as the domain and codomain).

My experiences learning and teaching about Adinkra symbols have broadened my views of the ways GeT students may reason about symmetry. I was first introduced to Adinkra symbols by a colleague at Portland State University, Dr. Joanna Bartlo, who had forwarded me the Culturally Situated Design Tools (CSDT) website (<https://csdt.org>) as a potential resource for simultaneously integrating computing (via Scratch) and mathematics from non-Eurocentric cultures into a Geometry for Middle School Teachers course I was teaching. I did not have time to incorporate the coding activity but instead asked my students to use the CSDT website to explore Adinkra symbols and their meanings, as a pre-assignment for class discussions of symmetry. Recalling my positive experiences teaching about automobile logos with my high school students, I believed learning about Adinkra would provide opportunities for students in my GeT class to form experiences connecting geometry to culture they could build on in their teaching.

The first two times I incorporated this assignment into my GeT courses, during remote teaching, I tasked students to record videos introducing to the class an Adinkra symbol they felt connected to, explain the symbol's meaning, identify its symmetries, and share why they chose to highlight it. I recall that subsequent class discussions often focused on statements of appreciation, as well as the absence of symmetry. Students corrected those who mistakenly attributed reflectional or rotational symmetries to Adinkra that were "not quite" correct.

It was not until the following term that I learned that this notion of "not quite symmetrical" was discussed on a different section of the CSDT website. In particular, I learned about the concept of *mutuality*, and I suggested it to the Transformations Working Group as a potential topic for us to explore how our students reasoned about symmetries, isometries, and their definitions. Determining the appropriate level of rigor and the appropriate properties and axioms for transformations in GeT courses have been part of ongoing discussions with the Transformation Working Group since its formation. We decided to embark on a lesson study to provide an opportunity for us to focus on our students' reasoning.

We ultimately decided to focus on an activity exploring Adinkra and mutuality because it provided our students (and us) with an opportunity to expand our knowledge about connections to mathematics from non-Eurocentric cultures. Furthermore, because "mutuality" is not a standard symmetry (i.e., described by rotation, reflection, or translation, or a composition thereof), and because it does not (yet!) have a commonly accepted mathematical definition, we saw an opportunity for students to experience genuinely open mathematical inquiry. (Boyce et al., 2021, para. 7)

It is important for GeT instructors to support prospective teachers' learning of geometry that is relevant to classroom teaching. Even though symmetries are not an explicit goal of the secondary curriculum, they provide opportunities to motivate transformational reasoning and extend students' understandings of connections between geometry and culture. Introducing mutuality is powerful because it encourages GeT students to reflect on the characteristics of their geometric reasoning, on what they are acting upon mentally when they conceive of symmetry, and how they might think about the absence of symmetry. It fosters an anti-deficit perspective of differences in others' reasoning and provides opportunities for prospective teachers to think about the role of cultural identity in teaching geometry. I hope readers of the newsletter check out the blog post (<https://blogs.ams.org/matheducation/2021/05/06/best-laid-co-plans-for-a-lesson-on-creating-a-mathematical-definition/>) and materials the Transformation Working Group has created and that we continue to build on our experiences learning and teaching about Adinkra and mutuality.

References

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GeT Course Student Learning Outcome #7

by Teaching GeT Working Group Members

Demonstrate knowledge of the history and basics of Euclid's *Elements* and its influence on math as a discipline.

Euclidean geometry is named after Euclid, the Greek mathematician who lived in Alexandria around 300 BCE. Euclid put together what was known at the time about Euclidean geometry into the thirteen books of *The Elements*. In *The Elements*, Euclid sets out a sequence of definitions, postulates (axioms for geometry), common notions (axioms common to all mathematical subjects), and propositions (theorems derived logically from the preceding materials). It is the “oldest extant large-scale deductive treatment of mathematics.” For most of the 2400 years since it was written, it was considered to be an essential text that any educated person would have studied; it is only in the last 150 years that this was no longer true. Likewise, for most of that time it was considered to be the gold standard of mathematical rigor; it is only in the last 150 years that its rigor has been surpassed.

Many mathematical disciplines can be illuminated by considering their histories, and this is especially true of geometry. Students need to know this history to place modern ideas about proof into context. A geometry class is likely where students will first encounter the notion of axiomatic proof, and for some of them it may be their only encounter with it. (See SLO 4 on Axioms, Theorems, and Models.) Euclid's first three postulates also set the stage for constructions using straightedge and compass. (See SLO 8 on Constructions.)

One particularly interesting piece of this history concerns Euclid's fifth Postulate: “That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.” Over the years, this postulate was considered inferior to the others, and many mathematicians tried to prove it from the others. It was only in the early 19th century that it was discovered that there is a geometry in which the other postulates are true, but this one is false. This is how hyperbolic geometry was discovered. (See SLO 9 on non-Euclidean geometry.)

Most students will not have seen Euclid's 5th postulate in its original form but would likely know something more like Playfair's postulate that given a line and a point not on the line that there is a unique line through the point parallel to the given line. Proving the logical equivalence to Euclid's 5th postulate is a good homework problem.

One question that remains relevant to modern geometry teachers is to what extent a geometry course should cover Euclid's actual writings and methods versus to what extent they should be replaced by simplified treatments (as is done in many high school geometry books) or by more technical rigorous methods (such as those based on Hilbert's *Foundations of Geometry*). As discussed above, for most of the time since it was written, almost all educated people studied *The Elements*. The debate about what treatments of geometry might be better than Euclid for modern students started in the mid-nineteenth century. There was enough debate that, in 1879, Charles Dodgson, the mathematician better known as Lewis Carroll, published a book entitled *Euclid and his Modern Rivals*, which argued that Euclid's treatment of geometry was superior for teaching students than any other treatment then proposed. This debate continues to this day.

One reason to consider teaching Euclid's original proofs, besides their historical interest, is that they are generally at the right level of sophistication for students; more modern treatments that are considered more rigorous might also be too complicated for students. One theory of cognitive development espoused by Jean Piaget among others states that “ontogeny recapitulates phylogeny,” that individual development often follows a similar path to historical development of a subject. Given the outsized role that Euclid's work has played in the historical development of geometry, it is not surprising that it might be at just the right level of sophistication for many students.


Looking at Euclid also makes it natural to consider where Euclid's treatment may have gaps. For example, Euclid's first postulate says that it is possible "to draw a straight line from any point to any point," but if you look at the way he uses it, he really assumes that the line drawn is unique. Looking at his proofs through the lens of a geometry where any two points can be connected but not necessarily uniquely, such as on the sphere, makes this issue immediately clear.

Euclid also introduces the idea of superposition to prove the Side-Angle-Side congruence condition for triangles. This argument was criticized because of the assumptions his proof makes about transformations without stating them explicitly. This can lead into a discussion of the transformations approach to Euclidean Geometry.

¹Wikipedia, "Elements." Retrieved January 27, 2022. Also found in other sources such as the preface to *The Delphi Collected Works of Euclid* (Delphi Classics, 2019).

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What can we learn from an assessment item about constructing perpendicular bisectors? Part 1: Problematizing the premise of the item

by Michael Weiss

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Introduction

Among the assessment items the GRIP team designed to probe students' knowledge of the student learning objectives (SLOs), item 15301 was intended to target SLO 3, Secondary Geometry Understanding: *Understand the ideas underlying the typical secondary geometry curriculum well enough to explain them to their own students and use them to inform their own teaching*. Assessment item 15301 asks:

Mr. Gómez taught students the usual procedure for constructing a perpendicular bisector for a segment. Veronica asked Mr. Gómez to explain why the construction works, meaning how they can be sure that the line that is constructed is indeed perpendicular to the segment and passes through the midpoint. How could Mr. Gómez explain that?

As is the case with many items intended to assess mathematical knowledge for teaching (MKT), this item is situated in the context of the work of teachers. As a consequence of this situated nature, the item actually consists of (at least) two different questions nested within each other. The first question is the question posed by Veronica to Mr. Gómez: "Why does the construction work... How can [we] be sure that the line that is constructed is indeed perpendicular to the segment and passes through the midpoint?" In what follows, I will refer to this as the *internal question*—the question that is asked *inside* the situation of teaching. The second question is the one posed to the GeT students themselves: "How could Mr. Gómez explain that?" I will refer to this as the *external question*—the question that is asked *about* the situation of teaching. These two questions are obviously closely related, but they do not necessarily have the same answer. The internal question is a purely mathematical question, one that calls for a mathematical explanation (which may or may not be a proof). In contrast, the external question is about *mathematics teaching* and (at least potentially) might call for a different kind of response. In the language of the MKT framework (Ball, Thames, & Phelps, 2008), we might say that the internal question probes common content knowledge (CCK), while the external question draws on other knowledge domains within MKT, such as (to name just two) knowledge of content and students (KCS) and knowledge of content and curriculum (KCC).

This description oversimplifies matters because the internal question is actually posed twice, in different terms. Initially Veronica asks "why the construction works"; almost immediately the question is explicated (by whom?) as meaning "how they can be sure that the

line that is constructed is indeed perpendicular to the segment and passes through the midpoint". However, this second question is more than a paraphrase or an elaboration of the first; it is an interpretation of it, one of many possible glosses of what Veronica might have meant by asking "why the construction works". Are there other plausible interpretations?

In addition to these layers, the internal and the external, let us consider a third, perhaps more fundamental, stratum of knowledge embedded in this assessment item: the level at which we ask "how *do* we construct a perpendicular bisector for a segment?" or, perhaps even more fundamentally, "what does it mean to construct a perpendicular bisector for a segment?" I will refer to these as *foundational questions*. Note that these foundational questions are presented by the item as settled matters; at the point at which the reader enters the situation of teaching, Mr. Gómez has already taught a method, one described as "the usual procedure." There is, evidently, no question what perpendicular bisectors are, what it means to "construct" a perpendicular bisector, nor how one does that. All of these are unproblematic for Mr. Gómez, for the item, and for the GeT student who is asked to respond to the item. It is only for Veronica (perhaps) that these are unsettled questions!

But perhaps Veronica is on to something. In what follows, I will attempt to problematize these foundational questions and show that they should not be taken entirely for granted. This *a priori* analysis will suggest a number of anticipated categories of response to the external question ("How could Mr. Gómez explain that?"), which in the second part of this paper I will use to categorize GeT student responses to item 15301.

Moving outward from this *a priori* analysis, we also can consider two additional, more empirical approaches to the assessment item, based on two different contexts in which the problem has been used. First, we may take note of the fact that this item has been used with GeT students, generating a collection of student responses. Second, we note that this item was one of many discussed by a collection of GeT instructors and others in the 2021 Summer GeT Assessment Item Workshop. What do GeT students say in response to this prompt, and what can we infer from those responses about their knowledge of geometry for teaching? Additionally, what did participants in the GeT summer assessment item workshop notice about the item and in the collected student responses?

I propose to describe item 15301 from five different points of view:

1. The foundational questions:
 1. What is a perpendicular bisector?
 2. What does it mean to construct a geometric object?
 3. Can we, in fact, construct a perpendicular bisector for any segment?
 4. How can we distinguish between a construction that actually works and one that only approximately works?
2. The internal questions:
 1. Why does the "usual construction" work?
 2. How can we be sure that the line constructed is indeed perpendicular to the segment and passes through the midpoint?
3. The external question:
 1. How could, or should, Mr. Gómez respond to Veronica?
4. Questions about GeT students:
 1. What do GeT students say in response to this prompt, and what can we infer from those responses about their knowledge of geometry for teaching?
5. Questions about GeT experts:
 1. What did participants in the summer workshop notice about the item and about student responses to the item?

My discussion of these questions will be divided into three articles. In the remainder of this article, I will discuss the foundational questions. In the second article in this series (see elsewhere in this issue), I will discuss the internal and external questions. A discussion of what item 15301 evoked from GeT students and experts will be contained in the third article in the series, to be published in a future issue of this newsletter.

Foundational questions about constructing perpendicular bisectors

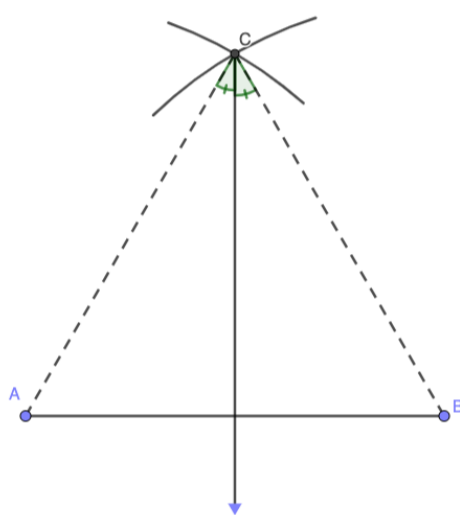
I here consider a number of basic questions regarding perpendicular bisectors and what it means to construct one.

A. What is a perpendicular bisector?

Given a line segment AB , a perpendicular bisector is a line that passes through the midpoint of AB at a right angle. In Euclidean geometry, every segment possesses a unique perpendicular bisector. This follows from two facts: (i) every segment possesses a unique midpoint; (ii) given any point P on AB , there exists a unique line perpendicular to AB through P .

It may be surprising that the phrase “perpendicular bisector” does not appear in Euclid’s *Elements* nor does Euclid provide a method for constructing a perpendicular bisector to a segment. To be sure, the *Elements* discusses perpendicular lines at great length, and presents methods for both constructing the midpoint of a segment (*Elements* 1.10) and constructing a perpendicular to a given segment through a point on that segment (*Elements* 1.11). Indeed, the construction of 1.10 locates the midpoint of a segment AB by means of a line drawn through AB at a right angle, but the fact that this line is perpendicular to AB is neither noted in the proof nor stated explicitly. Similarly the construction of 1.11 constructs a perpendicular to AB through a point C on AB by first constructing, on AB , a segment DE of which C is the midpoint, so that the perpendicular line eventually constructed does pass through the midpoint of DE —but once again this fact goes without notice. One might well argue that “perpendicular bisector” as a concept is tacitly present in these two items, but the tacitness is precisely the point I wish to focus on. Euclid sees no need to name, define, or describe a method of construction for “perpendicular bisectors.” They are simply not visible in the text. In fact, [a Google nGram search for the phrase “perpendicular bisector”](#) shows that the phrase did not begin to appear in English-language textbooks until the latter part of the 19th century.¹

Furthermore, even if we stipulate that *Elements* 1.10 and 1.11 are, at least implicitly, about constructing perpendicular bisectors, it is worth noting that the constructions presented there are not exactly the same as the “usual procedure” taught by Mr. Gómez in item 15301. Euclid’s construction of the midpoint for segment AB has essentially two steps: (i) construct an equilateral triangle ABC on segment AB , and (ii) bisect the vertex angle ACB . The point where the angle bisector intersects AB is its midpoint; this angle bisector also intersects AB at a right angle, although as mentioned above, Euclid does not note this latter fact. See the figure below.



It may be argued, contrary to what I have written above, that Euclid’s construction is not, in any essential way, different in practice from the “usual procedure” taught by Mr. Gómez because the performance of Euclid’s second step (bisect the vertex angle ACB) requires the performance of sub-steps that produce the bottom half of the figure constructed by the “usual procedure.” This is certainly true. But structurally, the two construction algorithms differ in that Euclid’s construction is described and conceptualized differently from the “usual” one. We observe that, in keeping with Euclid’s incrementalist style, this construction relies on two prior constructions: constructing an equilateral triangle on a given base (*Elements* 1.1) and bisecting an angle (*Elements* 1.9). Just as a theorem, once proved, may be cited as a reason in a proof of a subsequent theorem, so too may a construction, once demonstrated, be invoked as a single step in a subsequent construction. We might make an analogy with software coding; the prior constructions function as subroutines, which can be “called” by the main program. Once a subroutine has been compiled, we do not peer “inside” it to see the individual lines of code of that comprise it. The “usual” construction, in contrast, contains no subroutines; the algorithm is structurally “flat” with each operation performed within a single main routine. So even though the “usual” construction of a perpendicular bisector contains the same sequence of moves as the Euclidean construction of *Elements* 1.10, the two are organized and conceptualized in distinct ways.

B. What does it mean to construct a geometric object?

In Euclid’s *Elements*², a *construction* is a sequence of operations performed with a compass and an (unmarked) straightedge that, given some initial configuration of points and lines, produces a new configuration with some specified properties. Thus, for example, the first Proposition in the *Elements* is “to construct an equilateral triangle on a given finite straight-line.”³ In fact, the first three of Euclid’s five postulates are nothing more than statements of the basic operation of a compass and an unmarked straightedge.

In the *Elements*, constructions perform a function similar to what, in modern-day mathematics, we call an *existence proof*⁴. That is to say, if we wish to establish that there exists a line that (i) is parallel to any given line and (ii) passes through any specified point not on the line, the way we do that is by describing explicitly how such a parallel line can be constructed using a compass and straightedge (see *Elements* 1.31). In the absence of a construction, we have no way of knowing whether an object with a given set of properties exists at all, let alone whether it is unique. Euclid consistently avoids making use of any auxiliary objects in his proofs until the constructibility of the required objects has first been demonstrated. For example, Euclid’s proof of the Base Angles Theorem for

isosceles triangles (*Elements* 1.5) cannot make use of angle bisectors because angle bisectors are not constructed until later (*Elements* 1.9).

In a very real sense, an object that cannot be constructed using compass and straightedge does not exist in Euclidean geometry (or at least in the geometry of Euclid's *Elements*). For instance, because there is no construction algorithm for trisecting a general angle⁵, Euclid never considers what properties angle trisectors might possess or deploys them in arguments. They are simply outside of his universe of discourse.

Euclid's universe of discourse is determined by his choice of axioms; other axiomatic schemes for Euclidean geometry are possible. Beginning in the mid-20th century, many textbooks began replacing the compass and straightedge axiomatic structure with a different one, known as *ruler and protractor* axioms. This approach was pioneered by Birkhoff and Beatley in their 1941 text *Basic Geometry* and subsequently adopted by both the School Mathematics Study Group's 1960 textbook *Geometry* and by Moise and Down's popular 1964 text of the same name. As their names suggest, the ruler and protractor axioms take as fundamental the possibility of measuring the length of any segment and the magnitude of any angle⁶. In such a system, "constructing an object" means using a ruler and/or protractor to determine the placement of specific points and draw rays so that segments and angles have the required measurements. For example, in ruler and protractor geometry, trisecting an angle is trivial: the process is (i) measure the given angle, obtaining a real number x ; (ii) divide x by 3, obtaining another real number y ; (iii) draw a new angle with measure y .

This last example shows that answer to the question "What does it mean to construct a geometric object?" depends in a highly nontrivial way on the precise axiomatic framework in which we work. What counts as a construction in ruler and protractor geometry may not count as a valid construction in compass and straightedge geometry.

C. Can we, in fact, construct a perpendicular bisector for any segment?

As a practical matter, the answer to this question is no, or at least not exactly. Our ability to actually carry out an indicated construction algorithm is constrained not only by the mechanical qualities of the tools with which we work but also by the mechanical skills of the person wielding those tools.

Some of these constraints are fairly superficial and can be classified as "user error". When drawing a line with a straightedge or an arc with a compass, the user's hand may slip, the tool may slip across the paper, or the paper may slip on the table; any of these can result in "lines" that are not straight, "arcs" that wobble, and so forth. Then there is the issue of lining up the tip of the pencil with marks on the page while simultaneously aligning the edge of the ruler with a printed image; often the result is lines that are slightly offset from where they should be. Whether we regard these errors as inevitable or ascribe them to a lack of proficiency on the part of the user, they are common experiences that any classroom teacher should be aware of. So, when Veronica asks how they can be sure that the line that is constructed is indeed perpendicular to the segment and passes through the midpoint, one reasonable interpretation of her question might be simply: How do we reduce the amount of user error in the handling of the tools, so as to make sure that the algorithm produces an output that is what it ought to be?

Even if we imagine a student with a perfectly steady hand, there still remain constraints in the diagrammatic resources used to represent geometric objects—constraints that render any constructed figure only an approximation of what it is supposed to be. This is because the "points" and "lines" we draw (and draw on) are not really points and lines. *Points*, in Euclidean geometry, are understood as locations in space that have no spatial extent; they are located at a place, but do not occupy any space. In Euclid's famous formulation, a point is "that of which there is no part" (i.e., an indivisible *atomon* with no interior substructure). The points that appear in printed diagrams, in contrast, are small black circles of ink, with a diameter of perhaps half a millimeter. Similarly, a line is supposed to be a purely one-dimensional object, "a length without breadth" in Euclid's words, but a printed line is always made up of ink strokes that have some thickness and are (if viewed under a microscope) easily seen to be only approximations of the perfectly straight objects they are intended to signify. When we then consider that students may work with pens or pencils that are not particularly sharp or whose tips may break in the middle of a construction; with rulers whose "straight" edges may have nicks and dents; and with all the other imperfections that are endemic to well-worn tools, we see quickly that even under the best of circumstances, the most we can possibly hope for from a construction of a perpendicular bisector is a (picture of a) line that approximately passes through the midpoint of a (picture of a) line at an angle that measures approximately 90 degrees.

Whether these sources of error or imprecision should be regarded as significant or not depends on many contextual factors, **but one** thing that should be clear is that these types of concerns—concerns about the disconnect between the idealized mathematical object and the physical representation of the object—cannot be entirely allayed by appealing to the mathematical rigor of the algorithm itself. If we are concerned not with logical justification but rather with imprecision that results from faulty tools or faulty

execution, the only way to detect or mitigate that imprecision is through recourse to empiricism. That is to say, even if we know we have performed the algorithm correctly, we may still worry that the result is not sufficiently precise for our purposes (whatever those purposes may be); in that case the only way to check the accuracy of the result is by making measurements with a ruler and protractor.

What is also true is that we cannot escape the limitations of our material world by seeking refuge in digital representations. Even dynamic geometry software requires some degree of approximation; points are stored as a pair of coordinates, and numerical data when stored in a digital device can only be stored to a finite number of digits. Any software system, no matter how sophisticated, must inevitably approximate, and such approximations invariably lead to breakdowns in the ability to resolve the difference between very close quantities. For example, this can be demonstrated in GeoGebra as follows. First, define a numerical value n controlled by a slider. Next, define two points: $A = (0, 0)$ and $B = (10^{-n}, 10^{-n})$. Now type the commands **f = Line(A, B)** and **g = PerpendicularBisector(A, B)**. As long as n is a relatively small integer the software has no problem either computing the equations of the two indicated lines or drawing their graphs. But as soon as n gets too large (say $n > 8$), both lines disappear from the graph, and their equations display as undefined. Zooming in on the display shows that the two points A and B are still resolvable, with sufficient magnification, as distinct points, but the software can no longer calculate the slope of the line through AB nor of the line perpendicular to it. The construction requires a level of precision in excess of that possessed by the tool.

As soon as we recognize that dynamic geometry software is constrained by the precision with which it can store and display numerical values, we begin to question whether it actually shows us what we think it shows us. Do the angle bisectors of a triangle, constructed in a dynamic geometry diagram, *really* meet at a point, or do they merely intersect at points that are too close to each other to be resolved as distinct by the software? How can we possibly tell the difference? In the end, regardless of whether our tools are digital or mechanical, our only means of determining the accuracy of a construction is empirical, and empirical measurements are, inevitably and essentially, always approximate.

D. How can we distinguish between a construction that actually works and one that only approximately works?

We have seen that different axiomatizations of Euclidean geometry lead naturally to different notions of what constitutes a “valid construction.” We have also seen that, regardless of the axiomatization in play, a construction that has been validated on theoretical grounds—that is, one that can be “proved to work”—will nevertheless produce results that are only approximately correct once it is enacted with actual tools (whether mechanical or digital).

We now consider the reverse situation: what of constructions that cannot be validated using a proof but that nevertheless produce results that are approximately correct—or even correct to within the limits of empirical verification?

Consider, for example, the following construction protocol for “trisecting” an arbitrary $\angle ABC$:

1. Construct an arc, centered on the vertex B and intersecting the two sides of the angle BA and BC in points P and Q , respectively.
2. Subdivide secant PQ into three equal parts with points R and S .
3. Draw rays BR and BS .

Every step of this construction can be carried out using a compass and straightedge. However, the construction is not logically valid; it does not produce three angles of equal measure, each one-third the measure of the original $\angle ABC$. For small enough angles this algorithm produces results that are reasonably close to the desired ones. How small an angle is “small enough,” and how close is “reasonably close”? For example, if we begin with an angle $\angle ABC$ measuring 20° , the “trisectors” produced by this method determine angles whose measurements differ from the correct values by less than 1%, or about 0.06° , much too small to discern with the eye or to measure with a protractor. Even for an angle measuring 40° , the difference between what is intended and what is actually produced is less than half a degree, too small to measure with precision, especially given the error inherent in our measurement tools.

So does this construction “work” or not? Can we be sure, to paraphrase Veronica, that the rays constructed by this method actually divide the angle into three equal parts? The logician and the mathematician object strenuously to this claim; surely we are not satisfied with “almost right” constructions. But all constructions are, at best, “almost right,” at least when they are translated from abstract descriptions of idealized objects into concrete marks drawn on paper with physical tools wielded by imperfect humans. If empiricism urges us to be skeptical about logically valid constructions, does it not also push us to be tolerant of logically invalid constructions that produce results that are so close to correct as to be indistinguishable from accurate ones?

Conclusions

The above consideration of what I have called “foundational questions” is intended not only to describe, in some detail, the backdrop to Veronica’s question to Mr. Gómez in item 15301 but also to bring to the surface some of the categories of perception and appreciation that might be brought to bear on answering the remaining questions articulated in the introduction:

2. The internal questions: Why does the “usual construction” work? How can we be sure that the line constructed is indeed perpendicular to the segment and passes through the midpoint?
3. The external question: How could, or should, Mr. Gómez respond to Veronica?
4. Questions about GeT students: What do GeT students say in response to this prompt, and what can we infer from those responses about their knowledge of Geometry for teaching?
5. Questions about GeT experts: What did participants in the summer workshop notice about the item and about student responses to the item?

Some of the themes that have emerged from this discussion are (i) the role of constructions as existence proofs in Euclidean geometry, (ii) the way in which different axiomatizations of Euclidean geometry lead to different notions of what counts as a “valid construction,” (iii) that even within a particular axiomatic structure, what counts as the “usual” construction is a matter of convention, (iv) that whether a construction “works” may, depending on context, depend not only on the logical validity of the sequence of steps performed but also on the degree to which the final result matches our specifications of it when measured using realistic tools, and that (v) depending on how much accuracy we require in the final output, a valid construction may produce incorrect results (when performed by imperfect users of imperfect tools), while (vi) an invalid construction may produce results that are “correct enough” for our purposes. All of these observations serve to problematize the premise of item 15301: “Mr. Gómez taught students the usual procedure for constructing a perpendicular bisector for a segment...” In Part 2 of this article, I explore the internal and external questions contained within the item itself.

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¹The first appearance of this phrase in a text intended for the study of plane (i.e. Euclidean) geometry appears to have been in *The Elements of Plane Geometry for the Use of Schools and Colleges* (1868) by Richard P. Wright, although an earlier appearance is found in an 1825 trigonometry textbook by Thomas Luby. The 1868 text by Wright also contains the earliest appearance in print that I have found of the “usual method” for constructing perpendicular bisectors.

²Throughout this article, the translation used is that of Fitzpatrick (2007), based on the Greek text of J.L. Heiberg

³Euclid uses the phrase “straight line” (or sometimes “straight-line,” as here) to denote what in modern parlance would be called a “line segment.” Lines, in Euclid, are always finite in extent, though capable of extension to arbitrary lengths (and hence infinite in potential).

⁴More specifically, they correspond to what in modern mathematics would be called a constructive existence proof. Modern mathematics also includes nonconstructive existence proofs, which demonstrate the logical necessity of something existing without explicitly showing how it can be constructed.

⁵The nonconstructibility of a 20° angle, and hence the impossibility of formulating a trisection algorithm for angles in general, was not established until 1837. It is unclear (and probably unknowable) whether Euclid regarded the trisection problem as unsolvable or merely unsolved; in any case, because it was unsolved, he never refers to it at all.

⁶More precisely, the ruler postulate asserts the existence of a one-to-one correspondence between the set of equivalence classes of congruent segments and the set of positive real numbers, and the protractor postulate asserts the existence of a one-to-one correspondence between the set of equivalence classes of congruent angles and the set of real numbers greater than 0 and less than 180.

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What can we learn from an assessment item about constructing perpendicular bisectors? Part 2: Anticipating responses to the item

by Michael Weiss

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Introduction

Recall that GeT assessment item 15301 asks:

Mr. Gómez taught students the usual procedure for constructing a perpendicular bisector for a segment. Veronica asked Mr. Gómez to explain why the construction works, meaning how they can be sure that the line that is constructed is indeed perpendicular to the segment and passes through the midpoint. How could Mr. Gómez explain that?

As was discussed in the introduction to Part 1 of this article, because this item is situated in the practice of teaching, it contains multiple nested questions. First, there is the “internal question” posted by Veronica to Mr. Gómez; second, there is the “external question” posed to the GeT students themselves. In the first part of this article, I considered certain “foundational questions” about constructions: what it means to “construct” a geometric object and what we mean when we ask if a construction “works.” In Part 2, we consider the internal and external questions: Why does the construction work? How could Mr. Gómez respond to Veronica?

The internal question: How can we respond to Veronica?

We begin by placing ourselves in Mr. Gómez’s position and ask: How can we respond to Veronica? We start with isolating the internal question (i.e., the question posed inside the narrative framework of item 15301):

Veronica asked Mr. Gómez to explain why the construction works, meaning how they can be sure that the line that is constructed is indeed perpendicular to the segment and passes through the midpoint.

We observe that Veronica’s actual words are not provided but, instead, paraphrased; moreover, the initial paraphrase (“Why [does] the construction work [...]”) is then interpreted in another, more explicit formulation (“How can they be sure that the line tha

constructed is indeed perpendicular to the segment and passes through the midpoint [?]”). Although the second formulation is intended to be merely a clarification of the first, they are ostensibly different questions, potentially calling forth different kinds of responses:

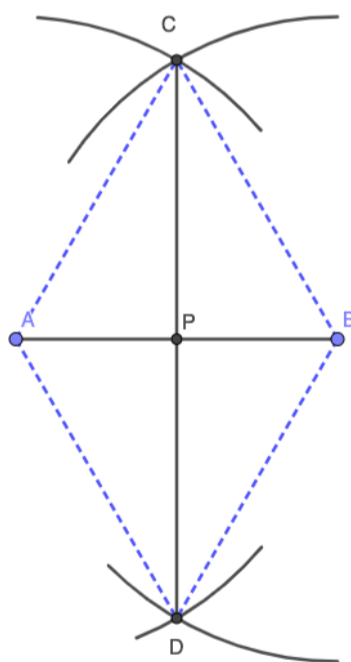
1. The first question calls for an *explanation* of why the construction “works” (whatever that means).
2. The second question calls for a *verification* that the result has the desired properties.

Although explanation and verification often go hand in hand, they are not the same thing (see de Villiers, 1990).

What kinds of responses should these two questions elicit? The second question reduces to the first if, and only if, we assume that a construction that is logically valid will, in fact, produce the desired outcome. This assumption, usually tacit in Geometry, suppresses concerns about precision and accuracy, and regards mathematical proof as both necessary and sufficient to ensure a correct product. But we need not make such an assumption, and we perhaps should not ascribe such a position to Veronica. It is entirely possible to recognize that a construction is logically valid and yet still have concerns that the product may be flawed due to errors in execution.

So we consider the two questions separately. The first question can be restated as: if we had a perfect compass, a perfectly steady hand, and were performing the construction process on an idealized diagram with perfectly small points and perfectly straight lines —if we imagine performing the construction in such a perfectly idealized world, free of human and mechanical error, how do we know that the result would indeed be a perpendicular bisector? Framed this way, it is clear that empirical considerations are neither relevant nor useful. The question is not about the actual construction performed on actual paper with an actual pencil but rather about a construction performed in the mind, an operation on generic conceptual objects, not on material objects. Verifying the accuracy of such an abstraction requires a similarly abstract validation system; that is, it requires a mathematical proof.

What could such a proof look like? A typical argument might begin by drawing segments from A and B to the two intersection points C and D , drawing segment CD , and letting P denote the intersection of CD with AB . The resulting figure (see below) then contains four small triangles ($\triangle ACP$, $\triangle BCP$, $\triangle DAP$, and $\triangle DBP$), sharing a common vertex at P ; the figure also contains four larger triangles ($\triangle ACB$, $\triangle ADB$, $\triangle CAD$, and $\triangle CBD$), each composed of two of the smaller triangles. The four small triangles are all congruent; this is evident from the symmetry of the figure. The four larger triangles are not all congruent, but they are congruent in pairs ($\triangle ACB$ with $\triangle ADB$ and $\triangle CAD$ with $\triangle CBD$). All of these congruences could, presumably, be proved using the given information and might potentially be useful in arriving at our goal. How do we navigate through this web of possibilities?



One plausible line of argument runs as follows: first, use the Side-Side-Side congruence criterion (SSS) to prove that the left-right pair of large triangles, $\triangle ACD$ and $\triangle BCD$, are congruent. Having established this, use the definition of congruent triangles (often called “CPCTC” for “Corresponding Parts of Congruent Triangles are Congruent”) to conclude that $\angle ACP$ and $\angle BCP$ are congruent. Now use the congruence of AC with BC , the congruence of $\angle ACP$ and $\angle BCP$, and the self-congruence of CP to conclude (by Side-Angle-Side) that $\triangle ACP$ is congruent to $\triangle BCP$. From this, we again use CPCTC to extract both the conclusion that $AP = BP$ (and thus that P is the midpoint of AB) and that $\angle APC$ is congruent to $\angle BPC$ (and therefore the four angles formed at P are all right angles).

Even in summary form, the proof is quite complex, as it requires first proving one pair of triangles is congruent, then using that fact (and consequences of it) to prove that a second pair of triangles is congruent. If this were written out in accordance with the usual norms of proving in secondary geometry, it would be even more complicated. (For example, using the fact that $\angle APC$ is congruent to $\angle BPC$ to infer that the angles at P are right angles may itself take several steps, depending on how “right angles” are defined.)

Is a different kind of proof available? Perhaps, depending on the axiomatic structure in use in Mr. Gómez's classroom. If, for example, his classroom makes use of a transformation-based geometry, the proof could be radically different (and perhaps quite a bit shorter). On the other hand, if the class is used to assigning Cartesian coordinates to the plane in order to write analytic proofs, the whole problem reduces to finding the coordinates of C, D, and P in terms of those of A and B.

The brief discussion above outlines some of the ways that the logical validity of the "usual" construction procedure for a perpendicular bisector can be established, responding to Veronica's initial question: "Why [does] the construction work [...?]" However, this would only satisfy Veronica if she, like the mathematician, regards logical validity as sufficient to ensure a correct output. If, instead, the re-paraphrase of Veronica's question ("How can they be sure that the line that is constructed is indeed perpendicular to the segment and passes through the midpoint [?]") signals a student who is worried that a valid construction may nevertheless produce inaccurate results due to human error or mechanical failure, a different kind of response may be in order; the way you make sure that the results come out correct is by (i) being very careful with your use of construction tools and (ii) checking the accuracy of the output of the algorithm with measurement tools after you are finished. Of course, our ability to check the accuracy of the output is inherently limited by the precision of our measurement tools, which can potentially work both to confer legitimacy where it does not belong (see the discussion of angle trisection in Part 1) and to cast doubts on a fundamentally sound method.

The external question: What should Mr. Gómez say?

We now turn to the external question of item 15301: What could, or should, Mr. Gómez say? This is the actual question of the item, the one directed to the GeT student whose knowledge we wish to assess.

We observe that this is a question of mathematical knowledge for teaching (MKT), one that draws, at least potentially, on multiple domains of knowledge (Ball, Thames, & Phelps, 2008), not only subject matter knowledge (SMK), as may be embodied in a mathematical proof, but also various aspects of pedagogical content knowledge (PCK). We list here just some of the domains of knowledge that Mr. Gómez may wish to or need to think about as he considers his response to Veronica:

1. Knowledge of Content and Curriculum (KCC)

Mr. Gómez may consider the fact that his classroom follows a particular curriculum, one in which the axioms and theorems of Geometry are arranged in a particular logical order. Knowing that other curricula exist, and are widely used, opens up a wide range of possibilities for his explanation, while the fact that one specific curriculum is in use in his classroom narrows down the possibilities. In a curriculum that is based on geometric transformations, the entire argument may come down to a single observation about the reflection symmetry in the construction. In a curriculum that consistently makes use of "analytic" methods (that is, the use of coordinates and algebra to represent geometric objects), an entirely different method would be appropriate.

Moreover the curriculum in Mr. Gómez's class is arranged in a specific sequence, one that may or may not correspond to the logical structure of the theory. For example, at some point in the course students are likely to learn that the diagonals of a rhombus are perpendicular bisectors of one another—a theorem whose proof is essentially identical to that of the construction under discussion. This theorem is a corollary of two others; the diagonals of any parallelogram bisect each other, and the diagonals of a kite are perpendicular. Since a rhombus is both a kite and a parallelogram, we immediately obtain the desired conclusion. When during the curriculum in use in Mr. Gómez's classroom are these properties of quadrilaterals taught? If they precede the current discussion of constructions, they could be appealed to as part of the response to Veronica. Whether this would be beneficial or not depends not only on Mr. Gómez's knowledge of the scope and sequence of topics in his curriculum but also on what he knows about Veronica.

2. Knowledge of Content and Students (KCS)

Knowledge of Content and Students (KCS) refers to a teacher's knowledge of how students may understand (or misunderstand) the content that is being taught—the conceptions and misconceptions they may hold. For example, many students fail to understand that a logical proof establishes the truth of a property for *all* figures satisfying the conditions of the problem; instead, they "view deductive proofs in geometry as proofs for a single case, the case that is pictured in the associated diagram" (Chazan, 1993, p. 362). Does this misconception underlie Veronica's question, "how can [we] be sure that the line that is constructed is indeed perpendicular to the segment and passes through the midpoint [?]" In other words, does her question stem from a concern that the construction, although proved valid for *this* figure, might not work for a different one? If so, Mr. Gómez's response should stress the generic nature of the argument, emphasizing words like "always" and "for every" in his argument.

On the other hand, perhaps Veronica, like many students, might be operating under a conception of “proof” rooted in empirical verification. If this were so, then Mr. Gómez might wish to try to reinforce a more deductive proof scheme, emphasizing that the purpose of a proof is to give us certainty without the need for measuring the results, while, perhaps, simultaneously reassuring her that if she wants to she can always measure the angles and segments, despite the fact that the result is a foregone conclusion.

3. *Knowledge of Content and Teaching* (KCT)

This domain of professional knowledge includes different ways of representing mathematical content. Some options that Mr. Gómez might wish to consider are:

- Different ways of representing a proof: for example, two-column form, paragraph proof, flowchart proof.
- Different ways of developing the proof with Veronica: as a presentation, as a Socratic dialogue, as an outline whose details are left for Veronica to fill in, etc.
- Different ways of representing the figure: on a chalkboard, on paper, in a dynamic geometry environment.

Related to the last of these, Mr. Gómez might also wish to consider the tools used to construct the figure. Should he draw the construction freehand? Should he use compass and straightedge? Should he have *Veronica* use the compass and straightedge? Would a DGS environment work better and if so, which one to use? None of these questions have a single correct answer, but Mr. Gómez must make decisions, whether consciously or not.

4. *Horizon Content Knowledge* (HCK)

Horizon Content Knowledge concerns Mr. Gómez’s knowledge of mathematics beyond the scope of the course he teaches. This might include questions like: does every segment have a unique perpendicular bisector in geometries other than that of the Euclidean plane? For example, in three-dimensional Euclidean geometry, there are infinitely many lines that are perpendicular bisectors to a given line; however, all of these lines lie on a single plane, so that the proper generalization of “perpendicular bisector” in three dimensions is a plane, not a line. In non-Euclidean geometry (for example, certain lattice geometries, as may be represented on a Geoboard), a perpendicular bisector may not exist at all because the “midpoint” of a segment may fall between two lattice points.

HCK may also include additional sources of knowledge, such as knowledge of the history of mathematics. Does Mr. Gómez know that Euclid’s *Elements* does not include “perpendicular bisector” as an object of study and that the construction of a midpoint to a given segment is organized differently than the “usual construction” he teaches his students? If so, does that knowledge suggest different ways he might respond to Veronica?

Does Mr. Gómez understand the role of constructions as furnishing “existence proof” in Geometry? That some objects cannot be constructed using compass and straightedge? That knowledge may elevate the significance of Veronica’s question; it is particularly important to know why *this* construction is possible because other constructions are *not*. The impossibility of trisecting an arbitrary angle, of doubling a cube, and of squaring a circle surely fall within the realm of HCK, and yet they can help motivate the significance of the constructions that are within the secondary Geometry course.

Concluding thoughts

In the discussion above, we complete our a priori analysis of GeT assessment item 15301. Having analyzed the foundational questions (What does it mean to construct? Is a construction possible?), the internal question posed by Veronica to Mr. Gómez (How do we know this construction works?), and the external question posed to the GeT students (What could Mr. Gómez say?), it remains to examine how GeT students responded to this item and what GeT instructors and other workshop participants saw in those responses. In the third and final part of this article, I will turn to those data sources and discuss what they suggest we can learn about GeT students’ knowledge of geometry for teaching and the usefulness of these assessment items for promoting discussion among practitioners.

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Transformations Working Group Update

by Julia St. Goar

In the Spring of 2022, the transformation group will likely focus on investigating existing teaching materials on geometry from a transformation perspective. As a reminder, a transformation perspective in geometry often involves approaching symmetry, congruence, similarity, and other geometry concepts/theorems using translations, rotations, reflections, and dilations. Through the investigation of such teaching materials, the group will explore the following:

- The group plans to focus on the affordances of different types of transformation lessons and activities.
- Such an investigation will likely further the group's on-going discussions about appropriate definitions and axiom systems that support a careful instructor treatment of concepts and proof in a transformation context.
- The group hopes to work towards addressing the following: Are there classic questions, classic exposures, and/or classic examples that students should have seen in a transformation context? What demonstrates the affordances of the transformation approach vs. a more "traditional" Euclidean approach? The resulting discussion may also contribute to a discussion with the Teaching GeT working group on the transformation SLO (Student Learning Objective).
- Some members have a particular interest in looking at how to use technology in a meaningful way with transformations.

The discussions on existing teaching material may involve continued reflection on teaching materials created by the transformation group in the Spring of 2021 (Boyce et. al., 2021). In the future, it is possible that the group may again embark on the creation of activities and lessons that support student understanding of transformations. As always, the group is open to new members and new ideas, so if you are interested, please consider joining.

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Teaching GeT Working Group Update

by Nathaniel Miller

The Teaching GeT working group has been keeping very busy. Several members of the working group got back from presenting our work developing essential Student Learning Objectives (SLOs) for GeT classes at the annual AMTE meeting in Las Vegas. We are aiming to have a website with our SLOs and narratives up by the end of the spring semester. Below, we have included the SL

along with one paragraph executive summaries taken from our submission to the AMTE Professional Book Series, Vol. 5: *Reflection on Past, Present and Future: Paving the Way for the Future of Mathematics Teacher Education*.

<p>SLO 1 [Proofs] Derive and explain geometric arguments and proofs.</p>	<p>Proof is a cornerstone of mathematics, and a GeT course should enhance a student's ability to read and write proofs of theorems, apply them, and explain them to others. Geometry offers one of the best opportunities for students to take ownership of proof in both the undergraduate and secondary curriculum. Because geometric proofs come with natural visualizations, they provide a rich environment for students to think deeply about the arguments that are being made and to make sense of each statement. Students should understand that proof is the means by which we demonstrate deductively whether a statement is true or false, and that geometric arguments may take many forms. They should understand that in some cases one type of proof may provide a more accessible or understandable argument than another.</p>
<p>SLO 2 [Critique Reasoning] Evaluate geometric arguments and approaches to solving problems.</p>	<p>Students should have opportunities not only to write proofs, but also to evaluate proofs and other types of reasoning. Geometry courses provide a natural setting for students to reflect on their own reasoning, share their reasoning with one another, and then critique the reasoning of their peers. If students only ever see correct arguments given by teachers and textbooks, they may not carefully evaluate the arguments because they assume they will always be correct. The ability to evaluate other people's arguments is an important real-world skill that is related to, but separate from, the skill of proof writing. Critiquing reasoning is a competency that needs to be practiced in order to improve it, and is an essential skill for future geometry teachers.</p>
<p>SLO 3 [Secondary Geometry Understanding] Understand the ideas underlying current secondary geometry content standards and use them to inform their own teaching.</p>	<p>Future secondary geometry teachers must deeply understand specialized content that is aligned to national and state secondary standards, know the best practices for teaching the content, and be able to reflect on their teaching. Due to limited time and instructors' varied preferences in content selection, it is not practical to suggest a list of geometry topics to be covered in a GeT course. Thus, the GeT course should focus on helping students understand essential ideas emphasized in secondary geometry standards and use them to inform their future teaching. GeT instructors should be able to incorporate teacher preparation standards into their course designs in a way that fits their teaching agenda and introduce the national and state curriculum standards to future teachers. In addition, a GeT course should foster the construction of pedagogical content knowledge by sharing teaching techniques and by engaging students in conversations about teaching geometry content.</p>
<p>SLO 4 [Axiomatic Systems] Understand the relationships between axioms, theorems, and different geometric models in which they hold.</p>	<p>Geometry courses are one of the few places where students have opportunities to engage explicitly with axiomatic systems. Mathematical theories can be developed from a small set of axioms with theorems proven from those axioms. However, GeT students may have limited prior experience working with axiomatic systems. Students in GeT courses should gradually develop the ability to: (a) recognize and communicate the distinction between axioms, definitions, and theorems, and describe how mathematical theories arise from them, (b) construct logical arguments within the constraints of an axiomatic system, and (c) understand the roles of geometric models such as the plane, the sphere, the hyperbolic plane, etc., in identifying which theorems can or cannot be proven from a given set of assumptions.</p>
<p>SLO 5 [Definitions] Understand the role of definitions in mathematical discourse.</p>	<p>In a geometry class, definitions can be a fruitful area for students to explore. Students can propose their own definitions for geometric objects and relationships, they can engage in class discussions about mathematical definitions versus vague descriptions, and they can compare and contrast definitions that refer to different properties. Determining whether and when definitions have equivalent meanings prepares prospective teachers for the varieties of geometric definitions they may encounter in teaching secondary geometry. Prospective geometry teachers should understand the role of precision in definitions for geometric terms and relationships, including understanding that some geometric terms and relationships must remain undefined. They should also understand that there are a variety of acceptable choices for some geometric definitions, and that these choices can influence the structure of proofs. For example, proving that two lines are parallel because they do not intersect can be very different from proving that they are parallel because they are everywhere equidistant.</p>
<p>SLO 6 [Technologies] Effectively use technologies to explore geometry and develop understanding of geometric relationships.</p>	<p>Geometry courses need to utilize technology to help develop geometric reasoning and deepen students' understanding of geometric concepts and relationships. Two types of technologies that are beneficial for teaching geometry are dynamic geometry environments (DGEs) and digital proof tools (DPTs). DGEs support student understanding by allowing students to explore properties of geometric figures dynamically, which provides advantages over using paper and pencil. These explorations help students generate their own conjectures, test their conjectures, and provide justification and understanding for theorems. DGEs have been an important tool for teaching geometry since the 1990s. DPTs are an emerging technology that provide students with interactive figures to manipulate and opportunities to practice writing proofs with immediate feedback.</p>

<p>SLO 7 [Euclid] Demonstrate knowledge of Euclidean geometry, including the history and basics of Euclid's <i>Elements</i> and its influence on math as a discipline.</p>	<p>The study of many mathematical subjects can be illuminated by looking at their histories, but this is especially true of geometry. Euclidean geometry is named after Euclid, the Greek mathematician who lived in Alexandria around 300 BC. Euclid systematized the knowledge of geometry and included it in thirteen books called <i>The Elements</i>. In <i>The Elements</i>, Euclid set out a sequence of Definitions, Postulates (axioms for geometry), Common Notions (axioms common to all mathematical subjects), and Propositions (theorems derived logically from the preceding materials). For most of the 2400 years since it was written, it was considered to be an essential text and the gold standard of mathematical rigor. Students need to know this history to place modern ideas about proof into context and to understand mathematics as a human endeavor.</p>
<p>SLO 8 [Constructions] Be able to carry out basic Euclidean constructions and to justify their correctness.</p>	<p>Traditional geometric constructions are those done exclusively with a compass and straightedge. However, the term can be used more generally to include other tools and manipulatives such as paper-folding (e.g., using origami or patty paper), dynamic geometry environments, or transparent mirrors (e.g., MIRAs). Constructions remain an essential part of Euclidean geometry and therefore of a GeT course. Constructions support the development of mathematical thinking in several essential ways: they provide a natural opportunity for making mathematical arguments, encourage the use of precise mathematical language in communication, impart a sense of where assumptions in building mathematical systems come from, open discussions of the historical development of geometry, especially the work of Euclid, and give future teachers experience with the curriculum they will be expected to teach.</p>
<p>SLO 9 [Non-Euclidean Geometries] Compare Euclidean geometry to other geometries such as hyperbolic or spherical geometry.</p>	<p>Just as visiting another country can offer one a richer perspective on their own culture, the study of non-Euclidean geometries can help students to develop a deeper understanding of Euclidean geometry. The term “non-Euclidean geometry” is interpreted broadly here, referring to any geometry different from Euclidean, including spherical, hyperbolic, incidence, and taxicab geometries, among others. The choice of which non-Euclidean geometries to consider might depend on the demands of a particular GeT course, but all non-Euclidean geometries offer rich opportunities to explore and visualize novel and engaging worlds. Learning the properties of non-Euclidean geometries puts preservice teachers in the position of their students who may be learning Euclidean geometry for the first time. Moreover, non-Euclidean geometries challenge student assumptions about what is “true” or “obvious” in Euclidean geometry (e.g. the shape of a parabola or the angle sum in a triangle). In this way, exploring non-Euclidean geometries naturally encourages questions of “Why?” and “How?” and supports the development of mathematical thinking, especially the need for justification.</p>
<p>SLO 10 [Transformations] Use transformations to explore definitions and theorems about congruence, similarity, and symmetry.</p>	<p>Two main types of transformations arise in GeT courses: isometries (also known as congruence transformations) and similarity transformations. Isometries include reflections, rotations, translations, and glide reflections; a similarity transformation is the composition of an isometry and a dilation. A GeT course can contain a dedicated unit on transformations, or transformational concepts can be integrated throughout the course (even to the extent of a purely axiomatic treatment). In order to enhance and facilitate prospective teacher learning, familiar function notation can be incorporated in an introduction to transformations. This may lead to a better understanding of a sequence of transformations as a composition. While some GeT courses may begin with an informal approach to the understanding of reflections and rotations and extend the concepts of symmetry to study geometric shapes using transformations, others may instead begin solely with sequences of reflections, which can be used to generate all other isometries of the plane.</p>
<p>Process Objectives</p>	<p>All GeT courses should give PSTs many chances to experience and develop proficiency with the mathematical process skills of problem solving, oral and written communication of mathematical ideas, and productive collaboration. They should also have opportunities to engage with the progression of geometric exploration followed by conjecturing and construction of arguments.</p>

The group has been meeting every other week, and we would encourage anyone who is interested to join us. Anyone who is interested in joining us or who has any questions about this work can contact me at nathaniel.miller@unco.edu.

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Get to know the community

Four questions with [Sharon Vestal](#), Associate Professor at South Dakota State University



- **What is special about your GeT course? In 2-3 sentences, describe your GeT course.**

The GeT course at South Dakota State University is unique in a couple of ways—it is actually called Geometry for Teachers, and it only includes students planning to teach secondary mathematics. My course includes both content and pedagogy, with a large emphasis on best practices for teaching geometry.

- **Who are your students?**

My students are typically sophomore or junior majors in our department, seeking a Mathematics with Teaching Specialization degree. Frequently students are taking the GeT course at the same time as they take our Logic, Sets, and Proof course. Since I teach both courses in the fall semester, many students have me as their instructor for 6 credits of mathematics.

- **What are you most interested in learning/achieving through participating with the GeT: A Pencil community?**

I was among the group to attend the conference in Michigan in June 2018 so I have had the privilege of being involved in this group since it began. One of the best parts of this group has been building relationships with colleagues teaching GeT courses at institutions across the country. These relationships have led to other opportunities and projects that have helped in my professional development.

The GeT: A Pencil community has helped me grow as a teacher and scholar and has provided me with a group of people who have similar beliefs and struggles. I find myself looking forward to our meetings as we have meaningful conversations, and we all listen with an open mind. These thoughtful meetings happen because of mutual respect.

- **What is your favorite book you have read in the last few years?**

I have read a lot of books on teaching in the past few years, but one of my favorites is Instant Relevance by Denis Sheeran. I coordinate an NSF Robert Noyce Scholarship Program so each summer we have a conference and bring in an outside speaker, usually an author of some book on teaching. We invited Denis to come during the 2017 summer conference, and I have continued to stay in touch and communicate with him. Instant Relevance is about using current events or trends to teach content in your classroom. For example, one of our sessions was on how we might be able to use fidget spinners in the classroom because they were popular at the time.

Did you get promoted? Win a grant? Have a baby? Buy a house? We would love to feature your news, whether professional or personal! Email us at GRIP@umich.edu.

Member News



Ruthmae Sears, Associate Professor at University of South Florida, was elected as a fellow of the American Association for the Advancement of Science (AAAS). You can read more about this recognition and her work [here](#).



Tuyin An, Assistant Professor at Georgia Southern University, has received tenure and has been promoted to Associate Professor, effective August 1, 2022.



Justin Dimmel, Assistant Professor at the University of Maine, received an [NSF CAREER award](#) to support research on how “movement-based interactions with spatial inscriptions” can support the teaching and learning of high school geometry. More on this [here](#).

Upcoming Seminars



Theo’s reinvention of the logic off conditional statements’ proofs rooted in set-based reasoning

Presented by Dr. Paul Dawkins

Friday, March 25, 2022 at 2:00 PM ET via Zoom



Exploring Contemporary Geometry for Secondary Teachers

Presented by Dr. Younggon Bae

Friday, April 8, 2022 at 2:00 PM ET via Zoom

To submit professional or personal news to be featured in the next newsletter, please fill out [this form](#).

GeT Support

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